

17. RANDI NATAF, Sur des questions d'agrégation en économétrie, *Publications de l'Institut de Statistique de l'Université de Paris* 2 (1953), 5-61.
18. VORUGANTI SOMASUNDARA RAO, Disaggregated demand and some aspects of the pure theory of international trade, Ph.D. dissertation, University of Minnesota, 1970.
19. PAUL A. SAMUELSON, Constancy of the marginal utility of income, in "Studies in Mathematical Economics and Econometrics, In Memory of Henry Schultz," (Oscar Lange, Francis McIntyre, and Theodore O. Yntema, Eds.), University of Chicago Press, Chicago, 1942, pp. 75-91.
20. PAUL A. SAMUELSON, "Foundations of Economic Analysis," Harvard U. P., Cambridge, Mass., 1947.
21. PAUL A. SAMUELSON, The problem of integrability in utility theory, *Economica*, N. S., 17 (1950), 355-385.
22. PAUL A. SAMUELSON, Prices of factors and goods in general equilibrium, *Rev. Econ. Stud.* 21 (1953), 1-20.
23. PAUL A. SAMUELSON, Social indifference curves, *Quart. J. Econ.* 70 (1956), 1-22.
24. H. THEIL, "Linear Aggregation of Economic Relations," North-Holland, Amsterdam, 1954.

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ÚTLÁNS

Fishery/Quadratic I.

Notes on the Volterra-Quadratic Fishery

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I. INTRODUCTION

Fisheries are not factories. A brief review of the fishery economics literature, for example, Gordon [5], Scott [9], Christy and Scott [3], and Smith [11], suggests particular properties of the ocean fishery which require tools of analysis beyond those supplied by elementary economic theory: an analysis of the fishery must take into account the biological nature of fundamental capital, fish; it must recognize the common property feature of the open sea fishery; it must allow that the fundamental capital is the subject of exploitation. These properties can, however, be captured by a simple model.

I shall first outline the model, then use it to examine some special cases, and close with some observations on the control of optimal exploitation paths.

2. THE VOLTERRA-QUADRATIC CONSTRAINT

Since Volterra [12], population biologists have paid particular attention to the special case where the fruits of predation (x) depend upon the multiplicative interaction of a predator population (a) and its prey population (b):

$$x = Aab,$$

where A is a constant denoting the efficiency, from the predator's point of view, of the predator-prey relationship. This is a special case of the

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well-known ACMS [1] production function with the elasticity of substitution equal to 1 (Cobb-Douglas), equal imputed shares and homogeneity of degree 2.

Similarly, the quadratic reproduction function is common in the literature. See, for example, Plourde [6]. In the absence of predation

$$db/dt = b = g(b) = Bb(\bar{b} - b),$$

where B is a constant denoting net reproductive efficiency. Maximum sustainable b is \bar{b} while maximum sustainable yield (MSY) is associated with $b = \bar{b}/2$ (where b is a maximum).

Taken together, these functions yield a particularly simple growth relation. In the presence of predation

$$b(a, b) = Bb(\bar{b} - b) - Aab.$$

This growth relation will be employed to explore the economics of biological resource exploitation. Special reference will be made to the fishery, but it is understood that the analysis is of more general usefulness in the analysis of economic exploitation of biological resources.

3. ECONOMIC EXPLOITATION

The Volterra-quadratic growth relation is recognized as a dynamic biological-technical constraint imposed by nature and technology on the fishing "effort" of men and their equipment (a). The dynamic nature of the control problem has been recognized before in connection with fishery resources. In particular, Quirk and Smith [8] have employed more general functional forms than I intend to introduce. The present effort is justified on the grounds that: Volterra-quadratic assumptions are frequent in the fisheries literature; particularly simple, interpretable results are easily obtained for use as benchmarks in more developed, realistic models; a special "nonuniqueness of flow" problem occurs, particularly in the neighborhood of equilibrium. Shell [10] and others have encountered a similar situation in macroeconomic models but, with the exception of Clark [4], it has not been treated in the resource control literature.

Letting the super-hat denote an exogenous variable the control problem is

Maximize:

$$J(a, b, t) = \int_0^T (TR - TC) e^{-\rho t} dt,$$

where

$$TR = \hat{p} \cdot x = \hat{p} \cdot Aab,$$

$$TC = \hat{w} \cdot a,$$

\hat{p} —market price of a fish on the quay,

\hat{w} —market price of non-fish inputs,

ρ —the "appropriate" discount rate,

$$T \leq \infty,$$

subject to:

(a) The biological-technical constraint. From above

$$b = Bb(\bar{b} - b) - Aab.$$

(b) The left-hand transversality condition

$$b(0) = b_0.$$

(c) The right-hand transversality condition

$$b(T) = b_T \quad \text{if } T < \infty,$$

$$\lim_{T \rightarrow \infty} e^{-\rho T} qb = 0 \quad \text{if } T = \infty,$$

where q is the (shadow) price of a fish in the sea, to be introduced presently.

Form the present value Hamiltonian function

$$H = e^{-\rho t} \{(TR - TC) + qb\}$$

which is interpreted as the present value of current profits ($TR - TC$) plus the (imputed) value, at the (shadow) price (q), of the current rate of investment in building up the stock of fish (b).

An economically meaningful, and mathematically interesting, problem must specify a finite upper limit to the fishing effort (a). Consequently, I specify for the moment that

$$a \in [0, \bar{a}].$$

A stricter requirement on the upper bound of a will be introduced presently.

Making the appropriate substitutions from above

$$H(a, b, q, t) = e^{-\rho t} \{\hat{p} \cdot Aab - \hat{w} \cdot a + q[Bb(\bar{b} - b) - Aab]\}.$$

In the control theoretic context

a —the control variable (a flow),

b —the state variable (a stock),

q —the dynamic multiplier,

Pontryagin [7] necessary conditions for a maximum are

- that a maximize H for all b, q, t ,
- that the decline of the present value shadow price ($q \exp - \rho t$) equal b 's contribution to H .
- that b obey the biological-technical constraint.

Proceeding in steps,

- Maximize H with respect to a

$$a(q, b) = \begin{cases} \bar{a} \\ 0 \\ \in [0, \bar{a}] \end{cases} \quad \text{if } q \begin{cases} < \\ > \\ = \end{cases} \frac{\hat{p} \cdot Ab - \hat{w}}{Ab}. \quad (1)$$

The optimal value of a is an upper semicontinuous correspondence in q and b .

The condition is easily interpreted. Consider the locus of points

$$q = (\hat{p} \cdot Ab - \hat{w})/Ab, \quad \text{where } a \in [0, \bar{a}].$$

Write

$$(\hat{p} - q) Ab = \hat{w}.$$

The expression Ab will be recognized as the marginal physical product of a , that is, x_a . The expression $(\hat{p} - q)$ is the net value of a fish: its market price if caught minus its shadow price if uncaught. Example: if a bird in the hand is worth two in the bush, then a fish on the quay is worth two in the sea, and $(\hat{p} - q)$ is $\frac{1}{2}\hat{p}$. The condition reads, simply: "If the value of a 's marginal product equals its price (\hat{w}) then $a \in [0, \bar{a}]$ is indeterminate." There is "flow without uniqueness." (See Shell [10].) Elsewhere, if q exceeds $(\hat{p} \cdot Ab - \hat{w})/Ab$, the value of a 's marginal product falls short of its price and a is set to its minimum value ($a = 0$), and vice versa, ($a = \bar{a}$).

$$(b) \quad (d/dt)(qe^{-\rho t}) = -H_b \quad \text{or} \quad \dot{q} = -[B(\bar{b} - 2b) - \rho]q - (\hat{p} - q)Aa. \quad (2)$$

Again, a straightforward interpretation is possible.

Write (2) as

$$(\hat{p} - q)Aa + \dot{q} = \rho q - B(\bar{b} - 2b)q.$$

The lhs is the value of the marginal product of a fish in the sea plus capital gains. It is the marginal net payoff from an uncaught fish. The rhs is the "financial cost" of an uncaught fish minus (plus) the value of "appreciation" (depreciation) at the "biological own rate of interest." Taken together, terms on the rhs are the marginal net cost of an uncaught fish.

- The biological-technical constraint must be observed:

$$b = Bb(\bar{b} - b) - Aab. \quad (3)$$

Equations (1), (2), and (3), along with the transversality conditions completely describe optimal exploitation trajectories. If (1) determines that $a = 0$ or \bar{a} , then (2) and (3) are trivially Lipschitzian differential equations defined over the two open sets separated by the common boundary $q = (\hat{p}Ab - \hat{w})/Ab$. However, I wish to assume that $\bar{a} > (A/B)(\bar{b} - b)$ so that, from (3), $b < 0$ when $a = \bar{a}$. Thus, there can be no singularity ($\dot{q} = b = 0$) in either open set.

If a singularity exists, it must be found along the common boundary. Set $\dot{q} = 0$ in (2), $b = 0$ in (3), and substitute out a . The result is

$$q(b) \Big|_{\substack{b=0 \\ \dot{q}=0}} = \hat{p}B(\bar{b} - b)/(\rho + Bb). \quad (4)$$

These are combinations of q and b such that $b = \dot{q} = 0$ for any value of a . They are depicted in Fig. 1 as the xx curve. But it is required, in addition, that $a \in [0, \bar{a}]$ as given by (1). That is,

$$q(b) \Big|_{a \in [0, \bar{a}]} = \frac{\hat{p} \cdot Ab - \hat{w}}{Ab}. \quad (5)$$

This curve appears as yy in Fig. 1. Where the two curves intersect at E , $\dot{q} = b = 0$ and $a \in [0, \bar{a}]$ is a^* , an equilibrium value of a associated with equilibrium values of b and q , denoted by b^* and q^* .

The value of b^* is given by (4) and (5).

$$\hat{p}B(\bar{b} - b^*)/(\rho + Bb^*) = (\hat{p}Ab^* - \hat{w})/Ab^*, \quad (6)$$

which is quadratic in b (unless $\rho = 0$ or $\hat{w}/A = 0$), due to an intersection of xx with the other branch of the yy curve (not illustrated) in the region where $q > p$ and $b < 0$.

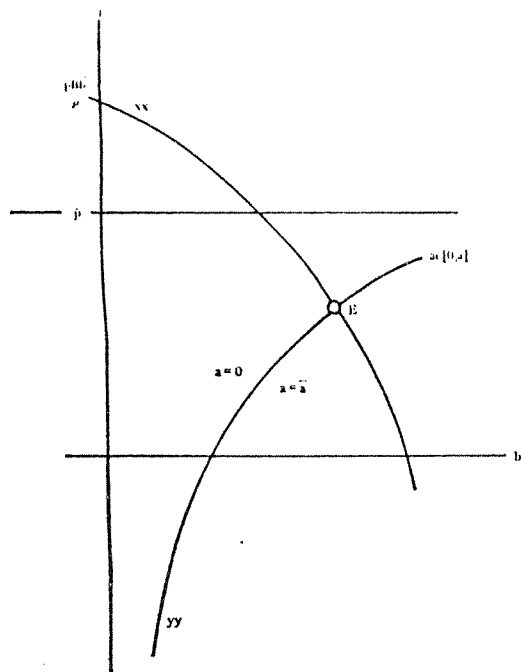


FIG. 1. Determination of equilibrium for the Volterra-quadratic fishery.

Similarly for q^*

$$(\hat{p}Bb - q^*\rho)/B(\hat{p} + q^*) = \hat{w}/A(\hat{p} - q^*). \quad (7)$$

I shall comment on these solutions in Section 6.

The complete dynamical system is depicted in Fig. 2, gathering together information in (1), (2), and (3), plus what has just been learned about a^* , b^* , and q^* . Several specimen trajectories (exploitation paths) are illustrated. The transversality conditions are used to select the one which is optimal.

The motions in phase space have the usual interpretations. Note, however, that above the yy line

$$b = Bb(\bar{b} - b) > 0$$

and below it

$$b = Bb(b - \bar{b}) - Aab < 0.$$

Thus E can always be reached in finite time. Also note that the stable,

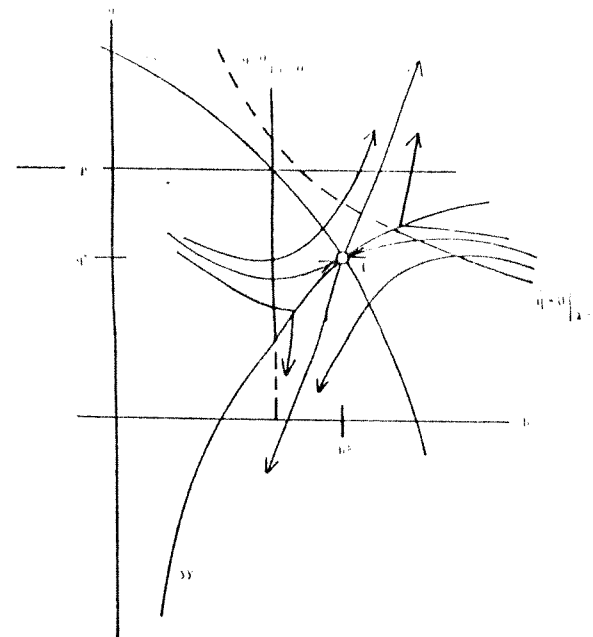


FIG. 2. Motions in phase space near the equilibrium.

arms leading toward E are not monotonic in q . I see no special significance in this observation, however.

4. SPECIAL CASES

A. The Cost of Exploitation Can Be Neglected.

This is the case examined by Plourde [6]. Equation (1) reduces to

$$a(q) = \begin{cases} \bar{a} \\ 0 \\ \in [0, \bar{a}] \end{cases} \quad \text{if } q \begin{cases} < \\ > \\ = \end{cases} \hat{p}. \quad (1')$$

The appropriate substitutions are made in (2) and (3). The correspondence no longer depends on b . The system is illustrated in Fig. 3.

This case is precisely analogous to the problem analyzed by Shell [10]. In his macroeconomic context, E represented a modified Golden Rule-Golden Age turnpike where the net rate of discount equaled the marginal

product of capital. An analogous interpretation can be made here. If $\hat{w}/A = 0$, then

$$q^* = \hat{p} \quad (5')$$

and

$$\rho = B(\bar{b} - 2b^*) \quad (6')$$

or

$$b^* = \bar{b}/2 - \rho/2B.$$

These results are consistent with Plourde's [6, p. 521].

Recall that $B(\bar{b} - 2b)$ is the biological own rate of interest, with a maximum at the point of extinction. Thus, deliberate extinction of the fishery will occur if the discount rate (ρ) exceeds the maximum own rate of interest ($B\bar{b}$). This is what one should expect if harvesting costs are zero. It is consistent with Clark's [4] findings, as well as Plourde's [6].

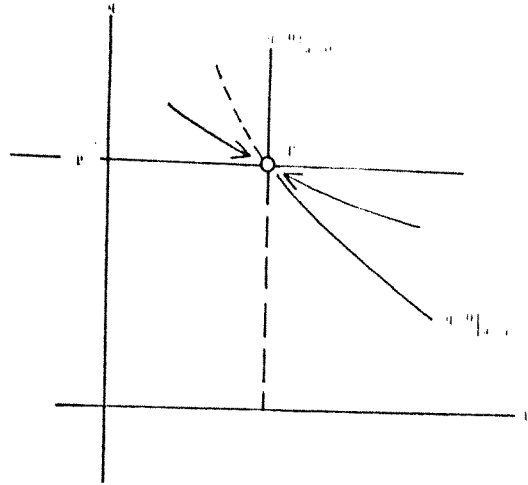


FIG. 3. Phase space when harvesting costs can be neglected.

B. The Rate of Discount (ρ) Is Zero

In this case

$$b^* = (1/2)(\hat{w}/A\hat{p}) + \bar{b}/2 \quad (6'')$$

and

$$q^* = \frac{\hat{p}\bar{b} - \hat{w}/A}{\hat{p}\bar{b} + \hat{w}/A} \hat{p}. \quad (7')$$

The presence of harvesting costs increases b^* beyond MSY. These costs ensure that q^* is less than \hat{p} . Even with no discounting, a fish is worth less in the sea than on the quay.

C. The Rate of Discount (ρ) is Infinite

Then

$$\hat{p}Ab^* = \hat{w}, \quad (6''')$$

$$q^* = 0. \quad (7'')$$

This is recognized as the classic rent dissipation solution for the unregulated, common property fishery with large numbers of firms. See, for example, Cheung [2].

Extinction will not occur as an equilibrium phenomenon. However, the dynamics of firm entry and exit along with the reproductive behavior of the fish stock may lead to "dynamic" extinction. See Smith [11, p. 188]. In any case, the equilibrium stock of fish is less than the socially optimal one.

D. The Production Function Can Be Written As $x = Aa(b+1)$ Instead of $x = Aab$

The new relation permits a positive catch at the point of extinction ($b = 0$). Following Clark [4], I seek conditions such that extinction of the exploited species will occur, even if the fishery is optimally managed.

The boundary of the two open sets is now expressed as

$$q = [\hat{p}A(b+1) - \hat{w}]/A(b+1),$$

while combinations of b and q such that $b = q = 0$ remain, as

$$q = \hat{p}B(\bar{b} - b)/(\rho + Bb).$$

Extinction occurs if these two equations solve for $b \leq 0$. Substituting out q yields the extinction condition

$$\hat{p}[(\rho - B\bar{b})/\rho] \geq \hat{w}/A.$$

Extinction is impossible if $B\bar{b}$ (b 's maximum biological own rate of interest) exceeds the discount rate. If it does not, extinction can occur if \hat{p} , ρ , and A are "large," relative to \hat{w} , B , and \bar{b} .

Again, the unregulated common property fishery can be simulated by letting $\rho \rightarrow \infty$. In that case extinction occurs if

$$\hat{p}A \cdot b > \hat{w}, \quad b = 1.$$

The condition is that the value of the marginal product of a exceed its price when b is the last remaining fish. A common property biological resource is doomed to extinction if it "pays" to wipe it out.

5. THE SYNTHESIZING FUNCTION

The solution to the control problem has so far yielded an open-loop control $a(t)$. It is well known that open-loop controls can give rise to doomsday paths if even small errors are made in assigning initial conditions, in timing, and the like. The control is "synthesized" if a corresponding closed-loop control $a(b)$ can be found. See Ref. [7, pp. 135-140].

The synthesis problem has not been solved in general, but the synthesized control is trivial in this case. It is

$$a(b) = \begin{cases} \bar{a} & \text{if } b > b^* \\ a^* & \text{if } b = b^* \\ 0 & \text{if } b < b^*. \end{cases}$$

The derivation is obvious from the discussion in Section 2, above.

The synthesizing function is illustrated in Fig. 4. Note that the control is discontinuous at $b = b^*$. The significance of this is discussed below.

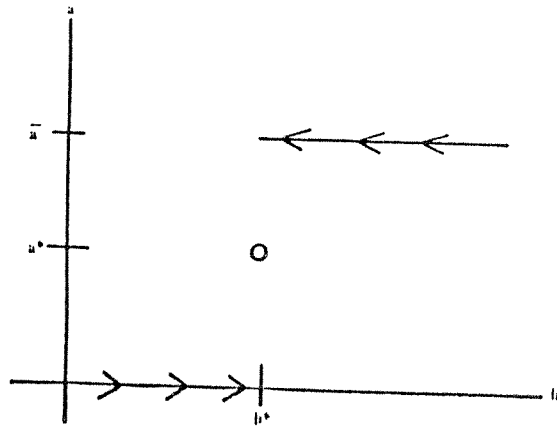


FIG. 4. The synthesized control of the Volterra-quadratic fishery.

6. CONTROL SCHEMES

Shadow prices, like q , are usually interpreted as the appropriate taxes or prices for correcting divergences between social and private costs and benefits. Such prices play a crucial role in decentralized control schemes.

In the present model, however, q is a useful conceptual device, but that is

about all. With the price of output fixed, and with Volterra production conditions, the total revenue function is linear in a . The total cost function is linear in a as well. That is why the synthesized control is discontinuous at b^* , reflecting "nonuniqueness of flow" encountered earlier. Quantitative control over a (or x) is absolutely required in the neighborhood of equilibrium.

By analogy, if two goods are perfect substitutes in production, quantitative controls are required to achieve nonzero output of both goods. Extreme linearities suggest extreme control devices.

I do not believe that most important fisheries exhibit such extreme linearities, although the Pacific Salmon fishery may come close to doing so. However, many fisheries may be sufficiently linear so that quantitative controls, as a practical matter, dominate price controls.

If this is correct, then it is important to know the value of b^* , and how it shifts when the underlying exogenous variables undergo change. From (6) one can determine that b^* rises if B or \hat{w} increase, and falls if A , p , or ρ increase. Actual determination of b^* is a nontrivial econobiometric exercise.

REFERENCES

1. K. J. ARROW *et al.*, Capital-labour substitution and economic efficiency, *Rev. Econ. Stats.* **43** (1961), 225-248.
2. S. N. S. CHEUNG, in "Economics of Fisheries Management - A Symposium," University of British Columbia, Institute of Animal Resource Ecology, Vancouver, 1970.
3. F. T. CHRISTY AND A. D. SCOTT, "The Common Wealth in Ocean Fisheries," Johns Hopkins, Baltimore, 1965.
4. C. W. CLARK, Profit maximization and the extinction of animal species, *J. Polit. Econ.* **81** (1973), 950-961.
5. H. S. GORDON, The economic theory of a common property resource, *J. Polit. Econ.* **62** (1954), 124-42.
6. C. G. PLOURDE, A simple model of replenishable natural resource exploitation, *Am. Econ. Rev.* **60** (1970), 518-521.
7. L. PONTRYAGIN *et al.*, "The Mathematical Theory of Optimal Processes," Interscience, New York, 1962.
8. J. P. QUIRK AND V. L. SMITH, in "Economics of Fisheries Management - A Symposium," University of British Columbia, Institute of Animal Resource Ecology, Vancouver, 1970.
9. A. D. SCOTT, The fishery: The objectives of sole ownership, *J. Polit. Econ.* **63** (1955), 116-24.
10. K. SHELL, Applications of Pontryagin's maximum principle to economics, in "Mathematical Systems Theory and Economics" (H. W. Kuhn and G. P. Szego, Eds.), Springer, Berlin, 1969.
11. V. L. SMITH, On models of commercial fishing, *J. Polit. Econ.* **77** (1969), 181-98.
12. V. VOLTERRA, "Theorie Mathématique de la Lutte Pour la Vie," Gauthier-Villars, Paris, 1931.